THE BAUM-CONNES CONJECTURE FOR KK-THEORY

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Abstract. We define and compare two bivariant generalizations of the topological K-group $K^{\mathrm{top}}(G)$ for a topological group G. We consider the Baum-Connes conjecture in this context and study its relation to the usual Baum-Connes conjecture.

0. Introduction

K-theory has been one of the most successful tools for analyzing C^*-algebras and C^*-dynamical systems. In this paper we consider the Baum-Connes conjecture, which proposes a way to compute the K-theory of a reduced crossed product algebra (see Section 2 for more details):

Conjecture 0.1 (The Baum-Connes Conjecture with Coefficients)**.** *Let G be a locally compact second-countable topological group. Then for any G-algebra B, the reduced assembly map*

$$\beta_{\mathrm{r}}^{B} : K_*^{\mathrm{top}}(G; B) \longrightarrow K_*(B \rtimes_{\mathrm{r}} G)$$

is an isomorphism.

If this is the case, we say that G satisfies the Baum-Connes conjecture for B.

Counterexamples to the Baum-Connes conjecture were constructed by Higson, Lafforgue and Skandalis, building on ideas of Gromov, in [HLS02]. Nonetheless, the conjecture for $B = \mathbb{C}$ still stands and has profound applications to geometry and algebra.

In order to study the KK-class of $B \rtimes_{\mathrm{r}} G$, we would like to generalize the conjecture to KK-theory. This would allow, in particular, to determine the mod-n K-theory of $B \rtimes_{\mathrm{r}} G$.

The formulation of the Baum-Connes conjecture (with coefficients) given in [BCH94, Conjecture 9.6] has a straightforward generalization to KK-theory (cf. Conjecture 2.3). However, one can easily see that while the right-hand-side of the conjecture is σ-additive in the first variable, the left-hand-side is *not*, in general. Hence this generalization of the conjecture to KK-theory fails for "trivial" reasons.

Meyer and Nest gave a reformulation of the Baum-Connes conjecture in [MN06, Theorem 5.2], using the notion of a Dirac morphism. Their approach yields another generalization of the conjecture to KK-theory (cf. Conjecture 3.17), which behaves better in many respects. We remark that this generalization also has well-understood counter-examples (cf. Example 3.10(2)), but we believe it still serves as a useful tool in the study of the KK-class of crossed product algebras.

In this paper, we compare the two approaches. In order to distinguish the two, we call the version based on [BCH94], Conjecture 2.3, the *naive Baum-Connes conjecture for KK-theory*, short for the naive generalization of the Baum-Connes conjecture to KK-theory and the version based on [MN06], Conjecture 3.17, simply, the *Baum-Connes conjecture for KK-theory.* We often omit the "for KK-theory" part.

Our main theorem is the following, see Theorem 4.5 for the precise statement.

Theorem 0.2. *Let B be a G-algebra. If the functor $KK_*(A, -)$ commutes with colimits, then the two generalizations of the Baum-Connes conjecture to KK-theory are equivalent for (A, B).*

If A satisfies the Universal Coefficient Theorem (cf. Theorem 5.1) and has finitely generated K-theory, then A satisfies the condition of Theorem 0.2 (cf. [RS87, Theorem 7.13]). A particular example is the dimension-drop algebra $\mathbb{I}_n$, $n \geq 2$, of (2.5). Since the mod-n K-theory of an algebra D can be computed as

$$K_*(D; \mathbb{Z}/n\mathbb{Z}) \cong KK_*(\mathbb{I}_n, D), \tag{0.1}$$

(see [DL96]), we can consider the (naive) Baum-Connes conjecture for $(\mathbb{I}_n, B)$ as a Baum-Connes conjecture for B in mod-n K-theory. It follows from Theorem 0.2, the two versions are equivalent. Moreover, they follow from the usual Baum-Connes conjecture:

Theorem 0.3 (Corollary 5.5 and Corollary 5.7)**.** *Let B be a G-algebra for which G satisfies the Baum-Connes conjecture (Conjecture 0.1). Then for any A satisfying the UCT, G satisfies the Baum-Connes conjecture for (A, B). If in addition, A has finitely generated K-theory, then G satisfies the* naive *Baum-Connes conjecture for (A, B).*

This is an immediate corollary of the treatment of UCT given in Section 5.

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1. Conventions

Throughout the paper, we assume that topological groups and topological spaces are *second-countable, locally compact and Hausdorff*, unless stated otherwise. Similarly, C^*-algebras are tacitly assumed to be *separable*, with the obvious exceptions such as multiplier algebras.

Let G be a topological group and let X be a topological space. A *G-algebra* is a C^*-algebra equipped with a strongly continuous action of G. If A is a G-algebra equipped with the trivial action of G, we often simply say that "A is a C^*-algebra". A $C_0(X)$-*algebra* is a C^*-algebra equipped with a $C_0(X)$-action, that is, a *nondegenerate* $*$-homomorphism from $C_0(X)$ to the central multipliers of the algebra. Here $C_0(X)$ denote the C^*-algebra of continuous functions on X vanishing at infinity.

Suppose that X is a *G-space*, that is, X is equipped with a continuous action of G. Then the algebra $C_0(X)$ is naturally a G-algebra via $(g \cdot f)(x) = f(g^{-1}x)$ for $g \in G$ and $f \in C_0(X)$. An *$X \rtimes G$-algebra* is a G-$C_0(X)$-algebra such that the action of $C_0(X)$ is *G-equivariant*.

We say that X is *G-compact* if the quotient X/G is compact and *proper* if the map $X \times G \to X \times X$, $(x, g) \mapsto (x, gx)$ is proper. Note that [BCH94] considers a slightly different notion of properness; see [BMP03, Bil04] for comparison. A G-algebra is said to be *proper* if it can be obtained from an $X \rtimes G$-algebra with X proper by forgetting the $C_0(X)$-action. Note that for a proper algebra the reduced and full crossed products coincide. If A is a G-algebra, following Kasparov, we write $A(X)$ for $C_0(X, A) = A \otimes C_0(X)$ and equip it with the diagonal action.

Let H be a closed subgroup of G. We denote the *restriction* (cf. [Kas88, Definition 3.1]) and the *induction* (cf. [Kas88, Theorem 3.5]) functors of Kasparov by $\mathrm{Res}_H^G : KK^{X \rtimes G} \to KK^{X \rtimes H}$ and $\mathrm{Ind}_H^G : KK^{X \rtimes H} \to KK^{X \rtimes G}$, respectively.

2. The Baum-Connes Conjecture for KK-theory, attempt 1

In this section, we consider the most simple-minded generalization of the Baum-Connes conjecture to KK-theory and show why this is not the desired one.

2.1. The naive Baum-Connes conjecture for KK-theory.

Let G be a topological group and let $\mathcal{E}G$ denote a *universal* proper space. (cf. [BCH94, BMP03, KS03]).

Definition 2.1. Let A be a C^*-algebra[1] and let B be a G-algebra. An inclusion $Y_1 \subseteq Y_2$ of G-compact subsets of $\mathcal{E}G$ induces a natural map

$$KK_*^G(A(Y_1), B) \to KK_*^G(A(Y_2), B). \tag{2.1}$$

We define the *naive topological KK-groups of* (A, B) as

$$KK_*^{\mathrm{naive}}(G; A, B) := \operatorname*{colim}_{\substack{Y \subseteq \mathcal{E}G \\ G\text{-compact}}} KK_*^G(A(Y), B), \quad * = 0, 1. \tag{2.2}$$

This is a straightforward generalization of the notion *topological K-group of B*: by definition, $K_*^{\mathrm{top}}(G; B) := KK_*^{\mathrm{naive}}(G; \mathbb{C}, B)$ ([BCH94, Definition 9.1], [BMP03, page 9]).

Any *proper G-compact* space Y gives rise to a canonical element

$$\lambda_{Y \rtimes G} \in K_0(C_0(Y) \rtimes G) = KK_0(\mathbb{C}, C_0(Y) \rtimes G) \tag{2.3}$$

by [KS03, page 178].

Definition 2.2. Let A be a C^*-algebra and let B be a G-algebra. The map

$$\beta_G^{A,B} : KK_*^{\mathrm{naive}}(G; A, B) \to KK_*(A, B \rtimes_{\mathrm{r}} G), \tag{2.4}$$

[1] Considered a G-algebra with the trivial action of G.

induced at the direct limit level by the composition

$$\begin{aligned}\beta_G^Y : KK_*^G(A(Y),B) &\xrightarrow{j_{\mathrm{r}}^G} KK_*(A(Y)\rtimes_{\mathrm{r}}G, B\rtimes_{\mathrm{r}}G)\\ &= KK_*(A\otimes(C_0(Y)\rtimes_{\mathrm{r}}G), B\rtimes_{\mathrm{r}}G) \xrightarrow{(1_A\otimes\lambda_{Y\rtimes G})\otimes} KK_*(A, B\rtimes_{\mathrm{r}}G),\end{aligned}$$

is called the (reduced) *naive assembly map* for (A,B). Here j_{r}^G denote the reduced *descent map* of Kasparov (cf. [Kas88, 3.11]).

Conjecture 2.3 (The naive Baum-Connes Conjecture in KK-theory)**.** *Let A be a C^*-algebra and let B be a G-algebra. We say that G satisfies the naive Baum-Connes conjecture for (A,B) if the naive assembly map $\beta_G^{A,B}$ is an* isomorphism *of abelian groups.*

The reason for the "naiveness" is that while the right-hand-side of the conjecture is σ-additive in the first variable, the left-hand-side is not. See Subsections 2.3 and 2.4 for more details.

Remark 2.4. (i) The original conjecture of Baum and Connes states that for any group the assembly map is an isomorphism for the pair $(\mathbb{C},\mathbb{C})$.
(ii) As stated in the introduction, counterexamples to the conjecture for $(\mathbb{C},B)$ were constructed by Higson, Lafforgue and Skandalis [HLS02].
(iii) Let

$$\mathbb{I}_n := \{f\in C([0,1],\mathbb{M}_n) \mid f(0)=0, f(1)\in\mathbb{C}I\}, \quad n\in\mathbb{Z}_{\geq 1}, \tag{2.5}$$

denote the n-th *dimension-drop algebra*. Then the mod-n K-theory can be computed (cf. [DL96]) by

$$K_*(D;\mathbb{Z}/n\mathbb{Z}) \cong KK_*(\mathbb{I}_n, D). \tag{2.6}$$

Thus the Baum-Connes conjecture for $(\mathbb{I}_n,B)$ can be considered as a Baum-Connes conjecture for B in mod-n K-theory.

Remark 2.5 (Nontrivial action on A)**.** Suppose that A is a G-algebra with a not necessarily trivial action of G. Then the topological KK-groups of (A,B) can be defined exactly as in Definition 2.1 and the definition of the assembly map can be modified to give an assembly map $KK_*^{\mathrm{naive}}(G;A,B)\to KK_*(A^G, B\rtimes_{\mathrm{r}}G)$, where A^G denote the fixed-point algebra of Kasparov [Kas88, Definition 3.2]. However, the right-hand-side "forgets" too much information about the action of G on A for the assembly map to be an isomorphism in general.

For instance, suppose that G is a finite group. Then $\mathcal{E}G=\{\mathrm{pt}\}$ and $KK_*^{\mathrm{naive}}(G;A,B) = KK_*^G(A,B)$ for any (A,B). Let H be an subgroup of G and let G act on G/H by left-translation. Then

$$KK_*^G(C(G/H),\mathbb{C}) \cong KK_*^H(\mathbb{C},\mathbb{C}) \cong KK_*(\mathbb{C}, C^*(H)), \tag{2.7}$$

by [CE01a, Proposition 5.14], whereas

$$KK_*(C(G/H)^G, \mathbb{C}\rtimes G) \cong KK_*(\mathbb{C}, C^*(G)), \tag{2.8}$$

since $C(G/H)^G \cong C(G\backslash G/H)\cong\mathbb{C}$. These can be quite different.

2.2. **Compact groups.** Let G be a *compact* group. Then $\mathcal{E}G = \{\mathrm{pt}\}$ and $\lambda_{\{\mathrm{pt}\}\rtimes G} \in K_0(C^*(G))$ is the class of the central projection in $C^*(G)$ corresponding to the *trivial* representation of G. Let A be a C^*-algebra (with the trivial G-action) and let B be a G-algebra. The topological KK-groups of (A, B) are simply the equivariant KK-groups:

$$KK_*^{\mathrm{naive}}(G; A, B) = KK_*^G(A, B)$$

and the assembly map equals the Green-Julg isomorphism

$$\beta_G^{A,B} : KK_*^G(A, B) \xrightarrow{\cong} KK_*(A, B \rtimes G). \tag{2.9}$$

See [Tu99, Proposition 6.25] for more details. Hence we have the following.

Proposition 2.6 (Green-Julg Isomorphism)**.** *Compact groups satisfy the naive Baum-Connes conjecture for any pair (A, B).* □

2.3. **σ-additivity.** In this subsection, we explain why the Conjecture 2.3 is called naive. We claim that we have a "problem", whenever we have a "nontrivial" colimit in the definition of the naive topological KK-group (cf. Definition 2.2).

Indeed, let A_i be C^*-algebras, $i \geq 1$. Then

$$KK_*(\oplus_i A_i, B \rtimes_{\mathrm{r}} G) \cong \prod_i KK_*(A_i, B \rtimes_{\mathrm{r}} G), \tag{2.10}$$

by the σ-additivity of KK in the first variable [Kas88, Theorem 2.9]. On the other hand,

$$KK_*^{\mathrm{naive}}(G; \oplus_i A_i, B) = \operatorname*{colim}_{\substack{Y \subseteq \mathcal{E}G \\ G\text{-compact}}} KK_*^G((\oplus_i A_i)(Y), B) \tag{2.11}$$

$$\cong \operatorname*{colim}_{\substack{Y \subseteq \mathcal{E}G \\ G\text{-compact}}} KK_*^G(\oplus_i A_i(Y), B) \tag{2.12}$$

$$\cong \operatorname*{colim}_{\substack{Y \subseteq \mathcal{E}G \\ G\text{-compact}}} \prod_i KK_*^G(A_i(Y), B), \tag{2.13}$$

again using [Kas88, Theorem 2.9]. But this is *not* necessarily isomorphic to

$$\prod_i KK_*^{\mathrm{naive}}(G; A_i, B) = \prod_i \operatorname*{colim}_{\substack{Y \subseteq \mathcal{E}G \\ G\text{-compact}}} KK_*^G(A_i(Y), B), \tag{2.14}$$

since limits and colimits do *not* commute in general. Hence we cannot expect G to satisfy the naive Baum-Connes conjecture for $(\oplus_i A_i, B)$ even if it does for (A_i, B) for all $i \geq 1$.

2.4. **Ascending union of open subgroups.** We give an explicit example illustrating the difficulties of 2.3.

Let A be a C^*-algebra with the trivial action of G and let B be a G-algebra.

Proposition 2.7 (cf. [BMP03, Theorem 5.1])**.** *Let H be an* open *subgroup of G. Then the inclusion of H in G determines a homorphism of abelian groups*

$$KK_*^{\mathrm{naive}}(H; A, B) \to KK_*^{\mathrm{naive}}(G; A, B).$$

If $G = \bigcup G_n$ is the union of ascending sequence of open subgroups

$$G_1 \subseteq G_2 \subseteq \cdots \subseteq G,$$

then

$$KK_*^{\mathrm{naive}}(G; A, B) \cong \operatorname*{colim}_{n\to\infty} KK_*^{\mathrm{naive}}(G_n; A, B). \tag{2.15}$$

Proof. The proof of [BMP03, Theorem 5.1] applies ad verbatim, once we notice that since the G-action on A is trivial, for any H-space X, we have

$$\operatorname{Ind}_H^G A(X) \cong A \otimes \operatorname{Ind}_H^G C_0(X),$$

where $\operatorname{Ind}_H^G : KK^H \to KK^G$ is the induction functor of Kasparov (cf. [Kas88, Theorem 3.5]). □

On the analytical side, we have the following.

Proposition 2.8 (cf. [BMP03, Theorem 4.1])**.** *Let H be an open subgroup of G, then canonical inclusion $C_c(H, B) \to C_c(G, B)$ extends to an injective $*$-homomorphism*

$$B \rtimes_{\mathrm{r}} H \to B \rtimes_{\mathrm{r}} G.$$

If $G = \bigcup G_n$ is the union of ascending sequence of open subgroups, then

$$B \rtimes_{\mathrm{r}} G \cong \operatorname*{colim}_{n\to\infty} B \rtimes_{\mathrm{r}} G_n. \tag{2.16}$$

Proof. See the proof of [BMP03, Theorem 4.1]. □

Definition 2.9. We say that a C^*-algebra A is KK-compact if $KK_*(A, -)$ is continuous, *i.e.* commutes with colimits.

Example 2.10.
1. If A satisfies the UCT (cf. Theorem 5.1) and has finitely generated K-theory, then A is KK-compact (cf. [RS87, Theorem 7.13]). In particular, the dimension-drop algebra $\mathbb{I}_n$ of (2.5) is KK-compact.
2. If A has a K-amenable Poincaré dual in the sense of [Con94, VI.4.β], then A is KK-compact.

Theorem 2.11 (cf. [BMP03, Theorem 6.3])**.** *Let A be a KK-compact C^*-algebra and let B be a G-algebra. Suppose that G is the union of ascending sequence of open subgroups G_n, each satisfying the naive Baum-Connes conjecture (2.3) for (A, B). Then G satisfies the naive Baum-Connes conjecture for (A, B).*

Proof. Since $KK_*(A, -)$ is continuous,

$$KK_*(A, B \rtimes_{\mathrm{r}} G) \cong \operatorname*{colim}_{n\to\infty} KK_*(A, B \rtimes_{\mathrm{r}} G_n).$$

Now the proof of [BMP03, Theorem 6.3] applies. □

Since KK-theory is *not* continuous in the second variable (cf. [Bla98, 19.7.2]), we *cannot* expect $KK_*(A, B \rtimes_{\mathrm{r}} G)$ to be isomorphic to $\operatorname{colim}_{n\to\infty} KK_*(A, B \rtimes_{\mathrm{r}} G_n)$ without restrictions on A. We demonstrate by example that the continuity of $KK_*(A, -)$ is necessary. This particular example was suggested by Nigel Higson (in the context of subsection 2.3).

Example 2.12. Let G denote the (discrete) abelian group $\bigoplus_{k\geq 1}\mathbb{Z}/2\mathbb{Z}$ and let $G_n \coloneqq \bigoplus_{k=1}^{n}\mathbb{Z}/2\mathbb{Z}$ considered as a subgroup of G. Then $G=\bigcup_{n\geq 1} G_n$. Note that abelian groups satisfy the Baum-Connes conjecture for any $(\mathbb{C}, B)$.

Let $A \coloneqq c_0(\Lambda)$ for some countable set Λ and let $B \coloneqq \mathbb{C}$. Then $B\rtimes G_n = C^*(G_n) \cong C^*(\mathbb{Z}/2\mathbb{Z})^{\otimes n} \cong (\mathbb{C}^2)^{\otimes n}$ and the inclusion map $B\rtimes G_n \to B\rtimes G_{n+1}$ is given by $f \mapsto f\otimes 1_{\mathbb{C}^2}$. Hence $KK(\mathbb{C}, C^*(G_n)) \cong (\mathbb{Z}^2)^{\otimes n} \cong \mathbb{Z}^{2^n}$ and the map induced by the inclusion is given by $\mathbb{Z}^{2^n} \ni p \mapsto (p,p) \in \mathbb{Z}^{2^{n+1}}$.

On the topological side, by Proposition 2.7,

$$\begin{aligned}
KK^{\mathrm{naive}}(G; c_0(\Lambda), \mathbb{C}) &= \operatorname*{colim}_{n\to\infty} KK^{\mathrm{naive}}(G_n; c_0(\Lambda), \mathbb{C}) \\
&= \operatorname*{colim}_{n\to\infty} KK(c_0(\Lambda), \mathbb{C}\rtimes G_n) \\
&= \operatorname*{colim}_{n\to\infty} \prod_{\Lambda} KK(\mathbb{C}, C^*(G_n)) \\
&= \operatorname*{colim}_{n\to\infty} \prod_{\Lambda} \mathbb{Z}^{2^n}.
\end{aligned}$$

On the analytical side, by Proposition 2.8,

$$\begin{aligned}
KK(c_0(\Lambda), \mathbb{C}\rtimes_{\mathrm{r}} G) &= \prod_{\Lambda} KK(\mathbb{C}, \mathbb{C}\rtimes G) \\
&= \prod_{\Lambda} \operatorname*{colim}_{n\to\infty} KK(\mathbb{C}, \mathbb{C}\rtimes G_n) \\
&= \prod_{\Lambda} \operatorname*{colim}_{n\to\infty} KK(\mathbb{C}, C^*(G_n)) \\
&= \prod_{\Lambda} \operatorname*{colim}_{n\to\infty} \mathbb{Z}^{2^n}.
\end{aligned}$$

Now it is a simple algebraic exercise to show that the two groups are different. Hence $G=\bigoplus_{k\geq 1}\mathbb{Z}/2\mathbb{Z}$ does *not* satisfy the naive Baum-Connes conjecture for $(A,B) = (c_0(\Lambda), \mathbb{C})$.

2.5. **Continuity.** Now we show that if A is KK-compact, then the naive Baum-Connes conjecture is stable under taking inductive limits of G-algebras. See Corollary 2.15 for the precise statement.

Lemma 2.13 (cf. [CEOO04, Subsection 1.1])**.** *Let A and B be G-algebras. Then*

$$\mathcal{F}_H(C_0(Y)) \coloneqq KK^H(\operatorname{Res}_G^H A(Y), \operatorname{Res}_G^H B), \quad H\in\mathcal{S}(G) \tag{2.17}$$

is a Going-Down functor in the sense of [CEOO04, Definitions 1.1].

If the G-action on A is trivial, then

$$\mathcal{F}^n(G) = KK_n^{\mathrm{naive}}(G; A, B). \tag{2.18}$$

Proof. The functor $\mathcal{F}_H^*$ is homotopy invariant by [Kas88, Proposition 2.5] and satisfies the Restriction axiom by [Kas88, Theorem 5.8]. Let

$$0\to C_0(U)\to C_0(Y)\to C_0(Y\backslash U)\to 0$$

be a short exact sequence of proper commutative H-algebras. Then it is equivariantly semi-split by [KS91, Corollary 6.2] and hence so is the sequence

$$0 \to A(U) \to A(Y) \to A(Y \backslash U) \to 0.$$

Then the same corollary implies that $\mathcal{F}_H^*$ is half-exact. Thus $\mathcal{F}$ satisfies the Cohomology axioms. Finally, by [Kas88, Lemma 3.6] there is a natural isomorphism:

$$A(\mathrm{Ind}_H^G(C_0(Y))) \cong \mathrm{Ind}_H^G(A(Y))$$

and [KS91, Remark 5.4] (see [CE01a, Proposition 5.14] for a slightly more general version) proves that $\mathcal{F}$ satisfies the Induction axiom. The last statement is clear. ☐

Often we omit the restriction functors from notation.

Lemma 2.14 (cf. [CE01a, Proposition 7.1])**.** *Let A be a KK-compact algebra. Then the functor $KK_*^{\mathrm{naive}}(G; A, -)$ is continuous.*

Proof. Let $B = \mathrm{colim}_{i\to\infty} B_i$ be a direct limit of G-algebras

$$\cdots \to B_i \to B_{i+1} \to \ldots.$$

For $H \in \mathcal{S}(G)$, set

$$\begin{aligned} \mathcal{F}_H^*(C_0(Y)) &= \operatorname*{colim}_{i\to\infty} KK_*^H(A(Y), B_i) \text{ and} \\ \mathcal{G}_H^*(C_0(Y)) &= KK_*^H(A(Y), B). \end{aligned}$$

Then $\mathcal{F}$ and $\mathcal{G}$ are Going-Down functors in the sense of [CEOO04, Definition 1.1] and the natural maps $\Lambda_H : \mathcal{F}_H \to \mathcal{G}_H$, induced by $B_i \to B$, form a Going-Down transformation in the sense of [CEOO04, Definition 1.3]. Moreover, in the notation of [CEOO04, Section 1],

$$\begin{aligned} \mathcal{F}^n(G) &= \operatorname*{colim}_{\substack{Y\subseteq \mathcal{E}G \\ G\text{-compact}}} \operatorname*{colim}_{i\to\infty} KK_n^G(A(Y), B_i) \\ &\cong \operatorname*{colim}_{i\to\infty} \operatorname*{colim}_{\substack{Y\subseteq \mathcal{E}G \\ G\text{-compact}}} KK_n^G(A(Y), B_i) \\ &\cong \operatorname*{colim}_{i\to\infty} KK_n^{\mathrm{naive}}(G; A, B_i) \qquad \text{and} \\ \mathcal{G}^n(G) &= KK_n^{\mathrm{naive}}(G; A, B). \end{aligned}$$

We need to show that $\Lambda^n(G) : \mathcal{F}^n(G) \to \mathcal{G}^n(G)$ is an isomorphism.

Let V be a finite dimensional Euclidean space equipped with a linear action of K. We have natural isomorphisms

$$KK^K(A(V), B_{(i)}) \cong KK^K(A, B_{(i)}(V)) \cong KK(A, B_i(V) \rtimes K) \tag{2.19}$$

by Kasparov's Bott periodicity theorem (cf. [CE01b, Lemma 7.7]) and the Green-Julg theorem (cf. Proposition 2.6). Since $B(V)\rtimes K \cong \mathrm{colim}_{i\to\infty}(B_i(V)\rtimes K$, the following

commutative diagram

$$\begin{tikzcd} \operatorname{colim}_{i\to\infty} KK^K_*(A(V),B_i) \arrow[r, "\Lambda_K"] \arrow[d, "\cong"] & KK^K_*(A(V),B) \arrow[d, "\cong"] \\ \operatorname{colim}_{i\to\infty} KK_*(A,B_i(V)\rtimes K) \arrow[r, "\cong"] & KK_*(A,B(V)\rtimes K) \end{tikzcd} \tag{2.20}$$

proves that the map $\Lambda_K : \mathcal{F}_K(C_0(V)) \to \mathcal{G}_K(C_0(V))$ is an isomorphism. Thus by [CEOO04, Theorem 1.4], $\Lambda^n(G)$ is an isomorphism and $KK^{\mathrm{naive}}(G;A,-)$ is continuous. □

Corollary 2.15 (cf. [CEN03, Proposition 2.5])**.** *Let A be a KK-compact algebra and let $\dots \to B_i \to B_{i+1} \to \dots$ be an inductive system of G-algebras. Suppose that either G is exact* or *all the connecting maps $B_i \to B_{i+1}$ are injective. Then if G satisfies the naive Baum-Connes conjecture for (A, B_i) for all i, then it satisfies for $(A, \operatorname{colim}_i B_i)$.* □

3. The Baum-Connes Conjecture for KK-theory, attempt 2

In this section, we consider an alternative generalization of the Baum-Connes conjecture to KK-theory. This generalization is already considered in [Kas88], in the case of almost connected groups.

3.1. Almost connected groups.

A topological group is said to be *almost connected* if its group of connected components is compact. The Baum-Connes conjecture for almost connected groups is known for the pair $(\mathbb{C}, \mathcal{K})$ with any action on $\mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space (cf. [CEN03]).

In this section, we study the naive Baum-Connes conjecture for an almost connected group G for a general pair (A, B). This will serve as a toy model and leads to the second approach to the Baum-Connes conjecture for KK-theory. The following two characteristics make it particularly nice to work with:

(a) it admits a G-compact universal proper space (hence difficulties from subsection 2.3 do not arise)
(b) it has a γ-element (cf. [CE01a, Definition 1.7]).

Let K be a maximal compact subgroup of G. Then the quotient $X := G/K$ equipped with the left-translation action of G is a universal proper G-space by [Abe75, Main Theorem]. Since X is G-compact, we have

$$KK^{\mathrm{naive}}_*(G;A,B) = KK^G_*(A(X),B). \tag{3.1}$$

Let $P = C_\tau(X)$ denote the graded algebra of the C_0-sections of the Clifford bundle on X and let $d = [d_X] \in KK^G_0(P, \mathbb{C})$ denote the Dirac element of X (cf. [Kas88, Definition-Lemma 4.2]).

Theorem 3.1. *Let G be an almost connected group and let A be a C^*-algebra and let B be a G-algebra. Then the assembly map $\beta^{A,B}_G$ can be identified with the "multiplication by the Dirac element"*

$$\otimes j^G_{\mathrm{r}}(1_B \otimes d) : KK_*(A, (B\otimes P)\rtimes_{\mathrm{r}} G) \to KK_*(A, B\rtimes_{\mathrm{r}} G)$$

via the commutative diagram

$$\begin{tikzcd} KK_*^{\mathrm{naive}}(G;A,B\otimes P) \arrow[r, "\beta_G^{A,B\otimes P}", "\cong"'] \arrow[d, "\otimes(1_B\otimes d)", "\cong"'] & KK_*(A,(B\otimes P)\rtimes_{\mathrm{r}}G) \arrow[d, "\otimes j_{\mathrm{r}}^G(1_B\otimes d)"] \\ KK_*^{\mathrm{naive}}(G;A,B) \arrow[r, "\beta_G^{A,B}"] & KK_*(A,B\rtimes_{\mathrm{r}}G). \end{tikzcd} \tag{3.2}$$

This is certainly well-known to the experts, but since we could not find any direct reference, we provide a proof (compare [Kas88, Theorem 5.10]). First we fix some notation.

Notation 3.2. Let G be a topological group and let H be a closed subgroup. For an H-algebra D, the canonical Morita equivalence from $D\rtimes_{\mathrm{r}}H$ to $(\mathrm{Ind}_H^G D)\rtimes_{\mathrm{r}}G$ is denoted by x_D (cf. [Kas88, Theorem 3.15]). For a G-algebra E, the canonical G-isomorphism from $E(G/H)$ to $\mathrm{Ind}_H^G \mathrm{Res}_G^H E$, given by $C_0(G/H,E) \ni f \mapsto [\widetilde{f} : g \mapsto gf(g^{-1}K)] \in \mathrm{Ind}_H^G \mathrm{Res}_G^H E$ is denoted by φ_E (cf. [Kas88, Lemma 3.6]).

Lemma 3.3 (cf. [CE01a, Proposition 2.3])**.** *Let G be an almost connected group and let $K\subseteq G$ be a maximal compact subgroup and let $X=G/K$. Let A be a C^*-algebra and let D be a K-algebra. Then the following diagram is commutative:*

$$\begin{tikzcd} KK_*^K(A,D) \arrow[r, "\beta_K^{A,D}"] \arrow[d, "\mathrm{Ind}_K^G"] & KK_*(A,D\rtimes K) \arrow[dd, "\cdot\otimes[x_D]"] \\ KK_*^G(\mathrm{Ind}_K^G A,\mathrm{Ind}_K^G D) \arrow[d, "{[\varphi_A]\otimes\cdot}"] & \\ KK_*^G(A(X),\mathrm{Ind}_K^G D) \arrow[r, "\beta_G^{A,\mathrm{Ind}_K^G D}"] & KK_*(A,(\mathrm{Ind}_K^G D)\rtimes_{\mathrm{r}}G). \end{tikzcd} \tag{3.3}$$

Proof. Take $x\in KK_*^K(A,D)$. Then we need to show that

$$(1_A\otimes\lambda_{X\rtimes G})\otimes j_{\mathrm{r}}^G([\varphi_A]\otimes \mathrm{Ind}_K^G x) = (1_A\otimes\lambda_{\{\mathrm{pt}\}\rtimes K})\otimes j_{\mathrm{r}}^K x\otimes[x_D]. \tag{3.4}$$

Since the action on A is trivial, $A\rtimes K\cong A\otimes\mathbb{C}\rtimes K$ and $\mathrm{Ind}_K^G A\cong A\otimes\mathrm{Ind}_K^G\mathbb{C}$ and under this identification $x_A=1_A\otimes x_{\mathbb{C}}$ and $\varphi_A=1_A\otimes\varphi_{\mathbb{C}}$. Thus, (3.4) is the consequence of the following identities:

(1) $\lambda_{X\rtimes G}\otimes j_{\mathrm{r}}^G[\varphi_{\mathbb{C}}]=\lambda_{\{\mathrm{pt}\}\rtimes K}\otimes[x_{\mathbb{C}}]$ (cf. [CE01a, (2.4)]) and
(2) $j_{\mathrm{r}}^K x\otimes[x_D]=[x_A]\otimes j_{\mathrm{r}}^G\,\mathrm{Ind}_K^G x$. (cf. [Kas88, Corollary 3.15]).

□

Lemma 3.4. *Let G be an almost connected group and let $K\subseteq G$ be a maximal compact subgroup. Let A be a C^*-algebra and let E be a G-algebra. Then the induction map*

$$KK_*^K(A,\mathrm{Res}_G^K E)\to KK_*^G(\mathrm{Ind}_K^G A,\mathrm{Ind}_K^G\mathrm{Res}_G^K E)$$

is an isomorphism.

See Corollary 3.11 for a stronger statement.

Proof. We keep the notation $X = G/K$. Consider the diagram,
(3.5)

$$\begin{tikzcd}
KK^K_*(A, \operatorname{Res}^K_G E) \arrow[rr, "\cdot\otimes 1_{C_0(X)}"] \arrow[dd, "\operatorname{Ind}^G_K"] & & KK^K_*(A(X), \operatorname{Res}^K_G E(X)) \\
& KK^G_*(A, E) \arrow[ul, "\operatorname{Res}^K_G"] \arrow[dr, "\cdot\otimes 1_{C_0(X)}"] & \\
KK^G_*(\operatorname{Ind}^G_K A, \operatorname{Ind}^G_K \operatorname{Res}^K_G E) \arrow[rr, "{[\varphi_A]\otimes\cdot\otimes[\varphi_E^{-1}]}"] & & KK^G_*(A(X), E(X)). \arrow[uu, "\operatorname{Res}^K_G"']
\end{tikzcd}$$

The lower-left corner is commutative by [Kas88, Theorem 3.6] and the upper-right corner is commutative by the functoriality of the restriction map. Moreover, the restriction map $\operatorname{Res}^K_G : KK^G_*(A, E) \to KK^K_*(A, \operatorname{Res}^K_G E)$ is surjective by [Kas88, Corollary 5.7], therefore the rectangle on the outside is commutative.

The classes $[\varphi_A]$ and $[\varphi_E]$ are equivalences by construction and the restriction $\operatorname{Res}^K_G :$ $KK^G_*(A(X), E(X)) \to KK^K_*(A(X), \operatorname{Res}^K_G E(X))$ is an isomorphism by [Kas88, Theorem 5.8]. It remains to show that $\cdot\otimes 1_{C_0(X)} : KK^K_*(A, D) \to KK^K_*(A(X), D(X))$ is an isomorphism. This follows from the equivariant Bott periodicity of Kasparov (cf. [CE01b, Lemma 7.7]) since, by [Abe75, Corollary A.6], X can be given the structure of a real vector space for which the action of K is linear. $\square$

Proof of Theorem 3.1. Commutativity of (3.2) follows from the multiplicative property of j^G_{r} (cf. [Kas88, Theorem 3.11]). The vertical map on the left

$$\otimes(1_B \otimes d) : KK^G_*(A(X), B\otimes P) \to KK^G_*(A(X), B)$$

is an isomorphism by [Kas88, Theorem 5.8], with inverse $\cdot\otimes(1_B\otimes\eta)$, where $\eta = \eta_X \in KK^G_0(\mathbb{C}, P)$ is the *dual-Dirac* element of Kasparov (cf. [Kas88, Definition-Lemma 5.1]). Finally, let C_V denote the Clifford algebra of the cotangent space to $X = G/K$ at $K \in X$ (cf. [Kas88, Theorem 5.10]). Then we have

$$P = \operatorname{Ind}^G_K \operatorname{Res}^K_G (C_V). \tag{3.6}$$

Combining Lemmas 3.3 and 3.4 with the Green-Julg isomorphism (Proposition 2.6), we see that the assembly map $\beta^{A, B\otimes P}_G$ is an isomorphism. This completes the proof. $\square$

Corollary 3.5. *Let G be an almost connected group and let A be a C^*-algebra and let B be a G-algebra. Then the assembly map gives an isomorphism*

$$KK^{\mathrm{naive}}(G; A, B) \cong KK(A, B\rtimes_{\mathrm{r}} G)\otimes j^G_{\mathrm{r}}(1_B\otimes\gamma), \tag{3.7}$$

where $\gamma = \gamma_G := \eta\otimes d \in KK^G(\mathbb{C}, \mathbb{C})$ is the γ-element of Kasparov (cf. [Kas88, Theorem 5.7]*).*

In particular, G satisfies the Baum-Connes conjecture for (A, B) if and only if $j^G_{\mathrm{r}}(1_B\otimes\gamma)$ acts as the identity on $KK(A, B\rtimes_{\mathrm{r}} G)$.

The right-hand-side of the expression is called the *γ-part* of $KK(A, B\rtimes_{\mathrm{r}} G)$.

Proof. This is a well-known argument. It follows from the proof of Theorem 3.1 that for any $x \in KK_*^G(A(X), B)$

$$\begin{aligned}\beta_G^{A,B}(x) &= \beta^{A,B\otimes P}(x\otimes(1_B\otimes\eta))\otimes j_{\mathrm{r}}^G(1_B\otimes d)\\ &= (1_A\otimes\lambda_{X\rtimes G})\otimes j_{\mathrm{r}}^G(x\otimes(1_B\otimes\eta))\otimes j_{\mathrm{r}}^G(1_B\otimes d)\\ &= (1_A\otimes\lambda_{X\rtimes G})\otimes j_{\mathrm{r}}^G(x)\otimes j_{\mathrm{r}}^G(1_B\otimes\eta d)\\ &= \beta_G^{A,B}(x)\otimes j_{\mathrm{r}}^G(1_B\otimes\gamma).\end{aligned}$$

The proof is completed using the identity $\gamma^2 = \gamma$. □

Remark 3.6. This corollary is not necessarily true for other groups with a γ-element in the sense of [CE01a, Definition 1.7], see Example 2.12.

3.2. **Strong Baum-Connes conjecture for almost connected groups.** Applying Yoneda's lemma to Theorem 3.1, we get the following.

Corollary 3.7. *Let G be an almost connected group and let B be a G-algebra. Then G satisfies the Baum-Connes conjecture for (A, B) for all C^*-algebras A if and only if $j_{\mathrm{r}}^G(1_B\otimes d) \in KK((B\otimes P)\rtimes_{\mathrm{r}}G, B\rtimes_{\mathrm{r}}G)$ is invertible if and only if $j_{\mathrm{r}}^G(1_B\otimes\gamma) = 1_{B\rtimes_{\mathrm{r}}G} \in KK(B\rtimes_{\mathrm{r}}G, B\rtimes_{\mathrm{r}}G)$.* □

If G and B satisfies the equivalent properties of Corollary 3.7, we say that G satisfies the *strong* Baum-Connes conjecture for B (cf. [MN06, Definition 9.1]).

Example 3.8. Any almost connected group with the Haagerup property satisfies $\gamma = 1 \in KK^G(\mathbb{C}, \mathbb{C})$ (cf. [HK01]), thus satisfies the strong Baum-Connes conjecture for any G-algebra B. Examples include $\mathrm{SO}(n, 1)$ and $\mathrm{SU}(n, 1)$.

Corollary 3.9 (cf. [MN06, Proposition 9.5])**.** *Let G be an almost connected group and let B be a type I G-algebra. Then G satisfies the strong Baum-Connes conjecture for B if and only if G satisfies the Baum-Connes conjecture for $(\mathbb{C}, B)$ and $B\rtimes_{\mathrm{r}}G$ satisfies the UCT.*

We include the short proof for the convenience of the reader.

Proof. First note that the algebra $(B\otimes P)\rtimes_{\mathrm{r}}G$ satisfies the UCT, since it is Morita equivalent to $\mathrm{Res}_G^K(B\otimes C_V)\rtimes K$, which is type I by Takesaki's theorem ([Tak67, Theorem 6.1]). Now suppose that G satisfies the strong Baum-Connes conjecture. Then, clearly, G satisfies the Baum-Connes conjecture for $(\mathbb{C}, B)$ and $B\rtimes_{\mathrm{r}}G$ satisfies the UCT by virtue of being KK-equivalent to $(B\otimes P)\rtimes_{\mathrm{r}}G$.

Conversely, suppose that G satisfies the Baum-Connes conjecture for $(\mathbb{C}, B)$ and $B\rtimes_{\mathrm{r}}G$ satisfies the UCT. Then by [Bla98, Proposition 23.10.1],

$$j_{\mathrm{r}}^G(1_B\otimes d) \in KK((B\otimes P)\rtimes_{\mathrm{r}}G, B\rtimes_{\mathrm{r}}G)$$

is invertible. □

Examples 3.10.
(1) Any almost connected group satisfies the strong Baum-Connes conjecture for $B = \mathcal{K}$. Indeed, let G be almost connected. Then G satisfies the Baum-Connes conjecture for $(\mathbb{C}, \mathcal{K})$ by [CEN03] and $\mathcal{K}\rtimes_{\mathrm{r}}G$ satisfies the UCT by [CEOO04, Proposition 5.1].

(2) Let Γ be a discrete subgroup of $\mathrm{Sp}(n,1)$ of finite covolume, $n \geq 2$. Then $B = \mathrm{Ind}_\Gamma^G \mathbb{C}$ is commutative (hence type I) but $(\mathrm{Ind}_\Gamma^G \mathbb{C}) \rtimes_{\mathrm{r}} G$ does not satisfy the UCT. Indeed by [Ska88, Corollaire 4.2], the algebra $C^*_{\mathrm{r}}\Gamma$, which is Morita equivalent to $(\mathrm{Ind}_\Gamma^G \mathbb{C}) \rtimes_{\mathrm{r}} G$, is not even KK-equivalent to a nuclear algebra, let alone an abelian one. Hence $\mathrm{Sp}(n,1)$ do *not* satisfy the strong Baum-Connes conjecture for $\mathrm{Ind}_\Gamma^G \mathbb{C}$. On the other hand, $\mathrm{Sp}(n,1)$ does satisfy the usual Baum-Connes conjecture for $(\mathbb{C}, B)$ for any B (cf. [Jul02]). This example is due to Skandalis [Ska88].

It follows from the equation (6.1) of [CE01a],

$$j_{\mathrm{r}}^G(1_{\mathrm{Ind}_K^G D} \otimes \gamma_G) = [x_D^{-1}] \otimes j_{\mathrm{r}}^K(1_D \otimes \gamma_K) \otimes [x_D] = 1_{\mathrm{Ind}_K^G D \rtimes_{\mathrm{r}} G}, \tag{3.8}$$

that G satisfies the strong Baum-Connes conjecture for the induced algebra $\mathrm{Ind}_K^G D$, for any K-algebra D (See also [MN06, Proposition 10.1]). This allows us improve on Lemma 3.4.

Corollary 3.11. *Let G be an almost connected group and let $K \subseteq G$ be a maximal compact subgroup. Let A be a C^*-algebra and let D be a K-algebra. Then the induction map*

$$\mathrm{Ind}_K^G : KK_*^K(A, D) \to KK_*^G(\mathrm{Ind}_K^G A, \mathrm{Ind}_K^G D)$$

is an isomorphism.

Proof. Every map except Ind_K^G in the commutative diagram (3.3) is an isomorphism, hence so is Ind_K^G. □

3.3. **The Baum-Connes conjecture for KK-theory.** When G is almost connected, Theorem 3.1 can be used to prove many nice properties of the assembly map. For the general case, we turn around everything, and reformulate the conjecture so that Theorem 3.1 becomes a tautology.

We recall some terminology from [MN06]. From now on, we work with equivariance with respect to transformation groupoids. This generality is needed in Section 4: we deduce Corollary 4.9, which is used in the proof of the Comparison Theorem 4.7, from the forgetful isomorphism of Theorem 4.8.

Let X be a G-space.

Definition 3.12 ([MN06, Definition 4.1])**.** An $X \rtimes G$-algebra is *compactly induced* if it is isomorphic to $\mathrm{Ind}_K^G D$ for some compact subgroup of $K \subseteq G$ and some K-algebra D. Let $\mathscr{CI} \subseteq KK^{X \rtimes G}$ denote the full subcategory of compactly induced algebras and let $\langle \mathscr{CI} \rangle$ denote the localizing subcategory generated by $\mathscr{CI}$. A morphism $f \in KK^{X \rtimes G}$ is called a *weak equivalence* if $\mathrm{Res}_G^K f \in KK^{X \rtimes K}$ is an isomorphism for all compact subgroups $K \subseteq G$.

Definition 3.13 ([MN06, Definition 4.5])**.** An element $d \in KK^{X \rtimes G}(P, C_0(X))$ is called a *Dirac morphism* for $X \rtimes G$ if d is a *$\mathscr{CI}$-simplicial approximation* of $\mathbb{C} \in KK^G$, that is,

(1) P is an object of $\langle \mathscr{CI} \rangle$ and
(2) d is a weak equivalence.

By [MN06, Proposition 4.6], Dirac morphisms exist, uniquely up to isomorphism, for any transformation groupoid. It follows that $\langle \mathscr{C}\mathscr{I} \rangle$ is a *coreflective* subcategory of $KK^{X\rtimes G}$.

Example 3.14. Let G be an almost connected group and let K be a maximal compact subgroup. Then the Dirac element $d = d_{G/K} \in KK^G(P, \mathbb{C})$ of [Kas88, Definition-Lemma 4.2] is a Dirac morphism for G in the sense of Definition 3.13 (Strictly speaking we need to replace P by an ungraded algebra.)

Let $d \in KK^{X\rtimes G}(P, C_0(X))$ be a Dirac morphism for $X\rtimes G$.

Definition 3.15. Let A be a C^*-algebra. and let B be an $X\rtimes G$-algebra. We define the *topological KK-group* of (A, B) as

$$KK^{\mathrm{top}}_*(X\rtimes G; A, B) := KK_*(A, (B\otimes_X P)\rtimes_{\mathrm{r}} G) \tag{3.9}$$

and the (reduced) *assembly map* as

$$\mu^{A,B}_{X\rtimes G} := \cdot\otimes j^G_{\mathrm{r}}(1_B\otimes_X d) : KK^{\mathrm{top}}_*(X\rtimes G; A, B) \to KK_*(A, B\rtimes_{\mathrm{r}} G). \tag{3.10}$$

Theorem 5.2 of [MN06] shows that this is indeed a generalization of the Baum-Connes conjecture. We write $K^{\mathrm{top}}_*(X\rtimes G; B)$ for $KK^{\mathrm{top}}_*(X\rtimes G; \mathbb{C}, B)$.

Remark 3.16. By [MN06, Lemma 5.1], if $d \in KK^G(P, \mathbb{C})$ is a Dirac morphism for G then $p^*_X(d) \in KK^{X\rtimes G}(P(X), C_0(X))$, where $p_X : X \to \{\mathrm{pt}\}$, is a Dirac morphism for $X\rtimes G$ and the natural identification

$$\mathcal{F} : B\otimes_X P(X) \cong B\otimes P \tag{3.11}$$

satisfies $\mu_{X\rtimes G} = \mu_G \circ \mathcal{F}_*$.

Conjecture 3.17. *(The Baum-Connes Conjecture in KK-theory). Let A be a C^*-algebra and let B be a G-algebra. We say that G satisfies the Baum-Connes conjecture for (A, B) if the assembly map $\mu^{A,B}_G$ is an isomorphism of abelian groups.*

This formulation doesn't have the shortcoming of the naive version, described in Subsection 2.3: If G satisfies the Baum-Connes conjecture for (A_i, B) for all i, then it satisfies for $(\oplus_i A_i, B)$.

4. Comparison of the two approaches

We know that the two formulations of the generalized Baum-Connes conjecture are not equivalent.

Example 4.1. Let $G := \bigoplus_{k\geq 1} \mathbb{Z}/2\mathbb{Z}$ and $A = c_0(\mathbb{Z})$ and $B = \mathbb{C}$. Then G satisfies Conjecture 3.17 for (A, B) by the σ-additivity of KK in the first variable, but not Conjecture 2.3 as demonstrated in Example 2.12.

4.1. **The comparison map.** First we generalize the naive topological KK-theory to transformation groupoids, following [Tu99] and [CEOO03]. Let X be a G-space.

Definition 4.2. Let A be a C^*-algebra and let B be an $X\rtimes G$-algebra. We define the naive topological KK-groups as

$$KK_*^{\mathrm{naive}}(X\rtimes G; A, B) := \operatorname*{colim}_{\substack{Y\subseteq X\times\mathcal{E}G\\ G\text{-compact}}} KK_*^{X\rtimes G}(A(Y), B). \tag{4.1}$$

As in [CEOO03, Section 1], there is a *forgetful* map

$$\mathcal{F} : KK_*^{\mathrm{naive}}(X\rtimes G; A, B) \to KK_*^{\mathrm{naive}}(G; A, B) \tag{4.2}$$

and an assembly map

$$\beta_{X\rtimes G}^{A,B} : KK_*^{\mathrm{naive}}(X\rtimes G; A, B) \to KK_*(A, B\rtimes_{\mathrm{r}} G) \tag{4.3}$$

satisfying $\beta_{X\rtimes G} = \beta_G \circ \mathcal{F}$, defined inductively via maps

$$\mathcal{F}_Y : KK^{X\rtimes G}(A(Y), B) \overset{F_X}{\to} KK^G(A(Y), B) \overset{(\pi_2|_Y)_*}{\to} KK^G(A(\pi_2(Y)), B) \tag{4.4}$$

$$\beta_{X\rtimes G}^{Y} : KK^{X\rtimes G}(A(Y), B) \overset{F_X}{\to} KK^G(A(Y), B) \overset{\beta_G^Y}{\to} KK(A, B\rtimes_{\mathrm{r}} G), \tag{4.5}$$

where $F_X : KK^{X\rtimes G} \to KK^G$ is the forgetful map and $\pi_2 : X\times\mathcal{E}G \to \mathcal{E}G$ is the projection onto the second coordinate.

Now we define a comparison map

$$\nu_{X\rtimes G}^{A,B} : KK_*^{\mathrm{naive}}(X\rtimes G; A, B) \to KK_*^{\mathrm{top}}(X\rtimes G; A, B). \tag{4.6}$$

Let $d \in KK^G(P, C_0(X))$ be a Dirac morphism for $X\rtimes G$. Then we have a commutative diagram

$$\begin{tikzcd} KK_*^{\mathrm{naive}}(X\rtimes G; A, B\otimes_X P) \arrow[r, "\beta_{X\rtimes G}^{A,B\otimes_X P}"] \arrow[d, "\cdot\otimes(1_B\otimes_X d)"] & KK_*^{\mathrm{top}}(X\rtimes G; A, B) \arrow[d, "\mu_{X\rtimes G}^{A,B}"] \\ KK_*^{\mathrm{naive}}(X\rtimes G; A, B) \arrow[r, "\beta_{X\rtimes G}^{A,B}"] & KK_*(A, B\rtimes_{\mathrm{r}} G). \end{tikzcd} \tag{4.7}$$

Lemma 4.3. *Let $x \in KK^{X\rtimes G}(B, B')$ be a weak equivalence and let A be a C^*-algebra. Then the natural map*

$$\cdot\otimes x : KK_*^{\mathrm{naive}}(X\rtimes G; A, B) \to KK_*^{\mathrm{naive}}(X\rtimes G; A, B')$$

an isomorphism.

Proof. Let $Y \subseteq X\times\mathcal{E}G$ be a G-compact subset. Then Y is proper and, by [MN06, Corollary 7.3], the algebra $A(Y)$ belongs to $\langle\mathscr{CI}\rangle$ (using G-compactness, we see that $A(Y)$ is contained in the triangulated subcategory generated by $\mathscr{CI}$). By [MN06, Proposition 4.4],

$$\cdot\otimes x : KK^{X\rtimes G}(A(Y), B) \to KK^{X\rtimes G}(A(Y), B') \tag{4.8}$$

is an isomorphism. □

As a corollary, the leftmost vertical map

$$\cdot\otimes(1_B\otimes_X d) : KK_*^{\text{naive}}(X\rtimes G; A, B\otimes_X P) \to KK_*^{\text{naive}}(X\rtimes G; A, B) \tag{4.9}$$

is an isomorphism.

Definition 4.4. We define the *comparison* map as the composition

$$\nu_{X\rtimes G}^{A,B} := \beta_{X\rtimes G}^{A,B\otimes_X P} \circ (\cdot\otimes(1_B\otimes_X d))^{-1}, \tag{4.10}$$

going from $KK_*^{\text{naive}}(X\rtimes G; A, B)$ to $KK_*^{\text{top}}(X\rtimes G; A, B)$.

This is an analogue of the map ν of [Tu99, Section 5]. It follows from the commutativity of (4.7) that

$$\mu_{X\rtimes G}^{A,B} \circ \nu_{X\rtimes G}^{A,B} = \beta_{X\rtimes G}^{A,B}. \tag{4.11}$$

Our main theorem is the following.

Theorem 4.5 (Comparison)**.** *Let A be a KK-compact algebra (cf. Definition 2.9) and B be an $X\rtimes G$-algebra. Then the comparison map $\nu_{X\rtimes G}^{A,B}$ is an isomorphism.*

The main difficulty in the proof is that we do not know if we can choose P, the source of the Dirac morphism, to be a proper algebra. However, this is the case for G almost connected and this fact turns out to be sufficient.

4.2. **Proof of the Comparison Theorem 4.5.** First we suppose that $X\rtimes G$ has a Dirac morphism $d \in KK^{X\rtimes G}(P, C_0(X))$ with P proper, that is, P admits a $C_0(X\times\mathcal{E}G)$-structure.

For any G-invariant subsets $V \subseteq Y \subseteq X\times\mathcal{E}G$ with V open and Y G-compact, we have the descent isomorphism of Kasparov and Skandalis

$$KK_*^{Y\rtimes G}(A(Y), B\otimes_X P_V) \cong KK_*(A, (B\otimes_X P_V)\rtimes G), \tag{4.12}$$

which is given by the forgetful map $F_Y : KK^{Y\rtimes G} \to KK^{X\rtimes G}$ followed by the assembly map $\beta_{X\rtimes G}^{Y}$ (cf. [Tu99, Proposition 6.25]). Here $P_V = C_0(V)P$ is the restriction to V.

Let $i_V : P_V \to P$ denote the inclusion and let $d_V = [i_V]\otimes d$. Then we have a natural map

(4.13)

$$KK_*(A, (B\otimes_X P_V)\rtimes G) \cong KK_*^{Y\rtimes G}(A(Y), B\otimes_X P_V) \overset{F_Y}{\to} KK_*^{X\rtimes G}(A(Y), B\otimes_X P_V)$$

$$\overset{\cdot\otimes d_V}{\to} KK_*^{X\rtimes G}(A(Y), B) \to KK_*^{\text{naive}}(X\rtimes G; A, B). \tag{4.14}$$

If A is KK-compact, then taking the colimit over V (cf. [Tu99, Proposition 5.7]), we get a map

$$\kappa_{X\rtimes G}^{A,B} : KK_*^{\text{top}}(X\rtimes G; A, B) = KK_*(A, (B\otimes_X P)\rtimes G) \to KK_*^{\text{naive}}(X\rtimes G; A, B). \tag{4.15}$$

It is clear that

$$\beta_{X\rtimes G}^{A,B} \circ \kappa_{X\rtimes G}^{A,B} = \mu_{X\rtimes G}^{A,B} \quad \text{and} \tag{4.16}$$

$$\nu_{X\rtimes G}^{A,B} \circ \kappa_{X\rtimes G}^{A,B} = \text{Id}. \tag{4.17}$$

Proposition 4.6. *Suppose that $X \rtimes G$ has a Dirac morphism $d \in KK^{X \rtimes G}(P, C_0(X))$ with P proper. Let A be a KK-compact algebra and let B be an $X \rtimes G$-algebra. Then the comparison map $\nu^{A,B}_{X \rtimes G}$ is an isomorphism, with inverse $\kappa^{A,B}_{X \rtimes G}$.*

Proof. We need to show that $\kappa^{A,B}_{X \rtimes G} \circ \nu^{A,B}_{X \rtimes G} = \mathrm{Id}$. By [MN06, Corollary 7.2], the element

$$p^*_{\mathcal{E}G}(d) \in KK^{(X \times \mathcal{E}G) \rtimes G}(P(\mathcal{E}G), C_0(X \times \mathcal{E}G)) \tag{4.18}$$

is invertible, where $p_{\mathcal{E}G} : X \times \mathcal{E}G \to X$ is projection onto the first coordinate. Let

$$\theta \in KK^{(X \times \mathcal{E}G) \rtimes G}(P(\mathcal{E}G), C_0(X \times \mathcal{E}G)) \tag{4.19}$$

denote the inverse. First we claim that the inverse of $\cdot \otimes (1_B \otimes_X d)$ is given by multiplication by θ on the left. More explicitly, let $j_Y : Y \subseteq X \times \mathcal{E}G$ be the inclusion of a G-compact subset and let $\theta_Y \coloneqq j^*_Y(\theta) \in KK^{Y \rtimes G}(C_0(Y), P \otimes_X C_0(Y))$. Let

$$F_Y : KK^{Y \rtimes G} \to KK^{X \rtimes G} \tag{4.20}$$

denote the forgetful map.

Let $x \in KK^{X \rtimes G}(A(Y), B)$. Then

$$\begin{aligned}
(\theta \otimes x) \otimes d &= (1_A \otimes F_Y \theta_Y \otimes_{A(Y) \otimes_X P} (x \otimes_X 1_P)) \otimes_{B \otimes_X P} (1_B \otimes_X d) \\
&= F_Y \theta_Y \otimes_{C_0(Y)} (x \otimes_X d) \qquad \text{(cf. [Tu99, Lemme 5.5])} \\
&= F_Y \theta_Y \otimes_{C_0(Y)} (d \otimes_X x) \\
&= F_Y \theta_Y \otimes_{C_0(Y)} (d \otimes_X 1_{C_0(Y)}) \otimes x \\
&= F_Y j^*_Y (\theta \otimes p^*_{\mathcal{E}G} d) \otimes x \\
&= x.
\end{aligned}$$

Now let $x \in KK^{X \rtimes G}(A(Y), B)$. Then $\nu^{A,B}_{X \rtimes G}(x) = \beta^{A, B \otimes_X P}_{X \rtimes G}(\theta \otimes x)$ and we need to write it in a form pluggable to $\kappa^{A,B}_{X \rtimes G}$.

The descent isomorphism and the continuity of K-theory imply that

$$KK^{Y \rtimes G}(C_0(Y), P \otimes_X C_0(Y)) \cong \operatorname*{colim}_V KK^{Y \rtimes G}(C_0(Y), P_V \otimes_X C_0(Y)). \tag{4.21}$$

Consequently, there exists $V \subseteq X \times \mathcal{E}G$ open and $\theta_{Y,V} \in KK^{Y \rtimes G}(C_0(Y), P_V \otimes_X C_0(Y))$ such that

$$\theta_Y = \theta_{Y,V} \otimes_{P_V} [i_V]. \tag{4.22}$$

Moreover, according to [Tu99, Proposition 5.12], there exists a G-compact subset $L \subseteq X \times \mathcal{E}G$, containing both V and Y, and $\theta' \in KK^{L \rtimes G}(C_0(L), P_V \otimes_X C_0(Y))$, where $C_0(L)$ acts on the first component of $P_V \otimes_X C_0(Y)$, such that

$$F_L \theta' = [j_{Y,L}] \otimes F_Y \theta_{Y,V} \tag{4.23}$$

in $KK^{X \rtimes G}(C_0(L), P_V \otimes_X C_0(Y))$, where $j_{Y,L} : Y \to L$ is the inclusion.

Then, as in [Tu99], we can write

$$\begin{aligned}\beta^{A,B\otimes_X P}_{X\rtimes G}(\theta\otimes x) &= \beta^Y(F_Y\theta_Y\otimes_{C_0(Y)}x)\\ &= \beta^Y((F_Y\theta_{Y,V}\otimes_{P_V}[i_V])\otimes_{C_0(Y)}x)\\ &= \beta^Y(F_Y\theta_{Y,V}\otimes_{C_0(Y)\otimes_X P_V}([i_V]\otimes_X x))\\ &= \beta^Y(F_Y\theta_{Y,V}\otimes_{C_0(Y)\otimes_X P_V}(x\otimes_X[i_V]))\\ &= \beta^Y(F_Y\theta_{Y,V}\otimes_{C_0(Y)}x)\otimes j^G[i_V]\\ &= \beta^L(F_L(\theta'\otimes_{C_0(Y)}x))\otimes j^G[i_V].\end{aligned}$$

It follows that

$$\begin{aligned}\kappa^{A,B}_{X\rtimes G}(\nu^{A,B}_{X\rtimes G}(x)) &= \kappa^{A,B}_{X\rtimes G}(\beta^{A,B\otimes_X P}_{X\rtimes G}(\theta\otimes x))\\ &= F_L(\theta'\otimes_{C_0(Y)}x)\otimes_{P_V}d_V\\ &= [j_{Y,L}]\otimes F_Y\theta_{Y,V}\otimes(x\otimes_X d_V)\\ &= [j_{Y,L}]\otimes F_Y\theta_{Y,V}\otimes[i_V]\otimes d\otimes x\\ &= [j_{Y,L}]\otimes(F_Y\theta_Y\otimes_P d)\otimes x\\ &= [j_{Y,L}]\otimes x.\end{aligned}$$

This completes the proof. □

If G is an almost connected group, then $X\rtimes G$ has a Dirac morphism with proper P. Thus, as in Corollary 3.5, we get the following.

Corollary 4.7. *Let G be an almost connected group. Let A be a KK-compact C^*-algebra and let B be an $X\rtimes G$-algebra. Then*

$$\beta^{A,B}_{X\rtimes G} : KK^{\mathrm{naive}}_*(X\rtimes G; A, B) \to KK_*(A, B\rtimes_{\mathrm{r}} G)$$

is an isomorphism onto the γ-part of $KK_*(A, B\rtimes_{\mathrm{r}} G)$. □

As a corollary, we get the following.

Theorem 4.8 (Forgetful Isomorphism, cf. [CEOO03, Theorem 0.1])**.** *Let A be a KK-compact algebra. Then the forgetful map*

$$\mathcal{F} : KK^{\mathrm{naive}}_*(X\rtimes G; A, B) \to KK^{\mathrm{naive}}_*(G; A, B)$$

of (4.2) is an isomorphism.

Proof. Corollary 4.7 implies that the theorem holds for G almost connected. Now the proof of [CEOO03, Theorem 0.1] applies. □

Corollary 4.9. *Let A be a KK-compact algebra. Then the assembly map $\beta^{A,B}_G$ is an isomorphism for $B \in \mathscr{CI}$.*

Proof. Proceeds as in [CEOO03, Section 4]. □

Now we are ready to prove the Comparison Theorem 4.5.

Proof of the Comparison Theorem 4.5. We need to show that $\nu^{A,B}_{X\rtimes G}$, or equivalently $\beta^{A,B\otimes_X P}_{X\rtimes G}$, is an isomorphism. By Theorem 4.8, it is enough to consider the case $X = \{\mathrm{pt}\}$. Let $\mathcal{BC}(G;A)$ denote the full subcategory of G-algebras $E \in KK^G$ such that $\beta^{A,E}_G$ is an isomorphism. Then $\mathcal{BC}(G;A)$ is clearly a triangulated subcategory of KK^G. Moreover, since A is KK-compact, $\mathcal{BC}(G;A)$ is closed under countable direct sums by Corollary 2.15 and contains $\mathscr{CI}$ by Corollary 4.9. Hence $\langle\mathscr{CI}\rangle \subseteq \mathcal{BC}(G;A)$. Now it is enough to notice that $B\otimes P$ belongs to $\langle\mathscr{CI}\rangle$ (cf. [MN06, Lemma 4.2]). □

5. The Universal Coefficient Theorem

In this section we develop a Universal Coefficient Theorem (UCT) for topological KK-functors and prove Theorem 0.3. As an application, we get an alternative proof of Theorem 4.5, in the case A satisfies the UCT and has finitely generated K-theory (such A's are KK-compact).

First we recall the UCT of Rosenberg and Schochet ([Bla98, Section IX.23]).

Theorem 5.1 (UCT [RS87])**.** *A C^*-algebra A is KK-equivalent to an abelian C^*-algebra if and only if it satisfies the UCT for every B, that is, there is a natural short exact sequence:*

$$\mathrm{Ext}^*_{\mathbb{Z}}(K_*(A),K_*(B)) \rightarrowtail KK_*(A,B) \twoheadrightarrow \mathrm{Hom}^*_{\mathbb{Z}}(K_*(A),K_*(B)). \tag{5.1}$$

□

In this situation, we simply say that A *satisfies the UCT*. The full subcategory of KK of algebras satisfying the UCT is the localizing subcategory $\langle\mathbb{C}\rangle \subset KK$ generated by $\mathbb{C}$ (cf. [MN06, Section 2.5]).

As in [CEOO04], we develop an abstract UCT first and specialize it to the topological KK-functors.

Definition 5.2. Let $\mathcal{C} \subseteq KK$ be a triangulated subcategory containing $\mathbb{C}$. A *UCT functor* on $\mathcal{C}$ is a cohomological functor $F : \mathcal{C} \to \mathbf{Ab}$, to the category of abelian groups, equipped with a zero-graded natural transformation

$$\gamma_A : F_*(A) \to \mathrm{Hom}^*_{\mathbb{Z}}(K_*(A), F_*(\mathbb{C})), \tag{5.2}$$

such that γ_A is an isomorphism whenever $K_*(A)$ is free and finitely generated. If, in addition, $\mathcal{C}$ is localizing and γ_A is an isomorphism whenever $K_*(A)$ is free, then we say that F is *σ-UCT*.

Proposition 5.3 (Abstract UCT)**.** *Let $\mathcal{C} \subseteq KK$ be a triangulated subcategory containing $\mathbb{C}$, and let F be a UCT functor on $\mathcal{C}$. Then for every C^*-algebra A in $\mathcal{C}$ with finitely generated K-theory, there is a natural short exact sequence, called the UCT exact sequence:*

$$\mathrm{Ext}^*_{\mathbb{Z}}(K_*(A),F_*(\mathbb{C})) \rightarrowtail F_*(A) \twoheadrightarrow \mathrm{Hom}^*_{\mathbb{Z}}(K_*(A),F_*(\mathbb{C})). \tag{5.3}$$

If F is σ-UCT, then the UCT exact sequence exists for all A in $\mathcal{C}$ (with no restriction on $K_*(A)$).

This is standard, but we include a proof here, because the proof of the usual UCT in [Bla98] uses an injective resolution of $K_*(B)$, whereas we use a free resolution of $K_*(A)$. As usual, it is enough to assume that F is defined only on $*$-homomorphisms, not arbitrary KK-morphisms.

Proof. We proceed as in [CEOO04, Section 3]. In both cases, it follows from Schochet's construction of the geometric resolution (cf. [Bla98, Proposition 23.5.1]) that there exists an algebra R in $\mathcal{C}$ and a $*$-homomorphism $\varphi : R \to A \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space, such that $K_*(R)$ is free and $\varphi_* : K_*(R) \to K_*(A \otimes \mathcal{K}) \cong K_*(A)$ is surjective. The rotated mapping cone triangle

$$\Sigma R \overset{-\Sigma\varphi}{\to} \Sigma(A \otimes \mathcal{K}) \to C_\varphi \overset{\varepsilon}{\to} R \tag{5.4}$$

is an exact triangle in $\mathcal{C}$. This gives a free resolution

$$0 \to K_*(C_\varphi) \overset{K_*(\varepsilon)}{\to} K_*(R) \to K_*(A) \to 0 \tag{5.5}$$

of $K_*(A)$ and consequently

$$\operatorname{Hom}_{\mathbb{Z}}(K_*(A), F_*(\mathbb{C})) \cong \ker \operatorname{Hom}_{\mathbb{Z}}(K_*(\varepsilon), F_*(\mathbb{C})) \qquad \text{and} \tag{5.6}$$

$$\operatorname{Ext}_{\mathbb{Z}}(K_*(A), F_*(\mathbb{C})) \cong \operatorname{coker} \operatorname{Hom}_{\mathbb{Z}}(K_*(\varepsilon), F_*(\mathbb{C})). \tag{5.7}$$

Moreover, since we have a commutative diagram

$$\begin{tikzcd} F_*(R) \arrow[r, "F_*(\varepsilon)"] \arrow[d, "\cong"', "\gamma_R"] & F_*(C_\varphi) \arrow[d, "\cong"', "\gamma_{C_\varphi}"] \\ \operatorname{Hom}_{\mathbb{Z}}(K_*(R), F_*(\mathbb{C})) \arrow[r] & \operatorname{Hom}_{\mathbb{Z}}(K_*(C_\varphi), F_*(\mathbb{C})), \end{tikzcd} \tag{5.8}$$

we may identify

$$\operatorname{Hom}_{\mathbb{Z}}(K_*(A), F_*(\mathbb{C})) \cong \ker F_*(\varepsilon) \qquad \text{and} \tag{5.9}$$

$$\operatorname{Ext}_{\mathbb{Z}}(K_*(A), F_*(\mathbb{C})) \cong \operatorname{coker} F_*(\varepsilon). \tag{5.10}$$

Finally, since F is a cohomological functor, we have a short exact sequence

$$0 \to \operatorname{coker} F_*(\varepsilon) \to F_*(\Sigma(A \otimes \mathcal{K})) \to \ker F_*(\Sigma\varepsilon) \to 0, \tag{5.11}$$

which in combination with the identifications (5.9) and (5.10) completes the proof. $\square$

For a fixed C^*-algebra B, the functor $A \mapsto KK(A, B)$ is a σ-UCT functor on $\langle \mathbb{C} \rangle$. Applying the Abstract UCT we get Theorem 5.1. As a corollary, we obtain the following.

Theorem 5.4. *Let B be a G-algebra. For any algebra A satisfying the UCT, we have the following natural short exact sequences and the assembly maps induce a map of short exact sequences*

$$\begin{tikzcd} \operatorname{Ext}^*_{\mathbb{Z}}(K_*(A), K^{\mathrm{top}}_*(G; B)) \arrow[r, tail] \arrow[d] & KK^{\mathrm{top}}_*(G; A, B) \arrow[r, two heads] \arrow[d] & \operatorname{Hom}^*_{\mathbb{Z}}(K_*(A), K^{\mathrm{top}}_*(G; B)) \arrow[d] \\ \operatorname{Ext}^*_{\mathbb{Z}}(K_*(A), K_*(B \rtimes_{\mathrm{r}} G)) \arrow[r, tail] & KK_*(A, B \rtimes_{\mathrm{r}} G) \arrow[r, two heads] & \operatorname{Hom}^*_{\mathbb{Z}}(K_*(A), K_*(B \rtimes_{\mathrm{r}} G)). \end{tikzcd}$$

Proof. Follows from the functoriality of the UCT sequence: the assembly map $\cdot\otimes j_{\mathrm{r}}^G(1_B\otimes d)$ induces a map of short exact sequences between the UCT sequences for $(A,(B\otimes P)\rtimes_{\mathrm{r}}G)$ and $(A,B\rtimes_{\mathrm{r}}G)$. □

Applying the Five-Lemma, we obtain the following.

Corollary 5.5. *Let B be a G-algebra. Suppose that G satisfies the Baum-Connes conjecture for $(\mathbb{C},B)$. Then for any algebra A satisfying the UCT, G satisfies the Baum-Connes conjecture for (A,B).* □

Next we consider the UCT for naive KK-theory.

Theorem 5.6. *Let B be a G-algebra. Then for any A satisfying the UCT and having finitely generated K-theory, we have the following natural short exact sequences and the assembly maps induce a map of short exact sequences*

$$\begin{tikzcd}
\mathrm{Ext}^*_{\mathbb{Z}}(K_*(A),K_*^{\mathrm{top}}(G;B)) \arrow[r, tail] \arrow[d] & KK_*^{\mathrm{naive}}(G;A,B) \arrow[r, two heads] \arrow[d] & \mathrm{Hom}^*_{\mathbb{Z}}(K_*(A),K_*^{\mathrm{top}}(G;B)) \arrow[d] \\
\mathrm{Ext}^*_{\mathbb{Z}}(K_*(A),K_*(B\rtimes_{\mathrm{r}}G)) \arrow[r, tail] & KK_*(A,B\rtimes_{\mathrm{r}}G) \arrow[r, two heads] & \mathrm{Hom}^*_{\mathbb{Z}}(K_*(A),K_*(B\rtimes_{\mathrm{r}}G)).
\end{tikzcd}$$

Proof. Let $\mathcal{C}$ denote the full subcategory of $\langle\mathbb{C}\rangle$ consisting of algebras with finitely generated K-theory. It is clear that $\mathcal{C}$ is a triangulated subcategory containing $\mathbb{C}$. Let B be a G-algebra. We consider the functor $F:\mathcal{C}\to\mathbf{Ab}$ given by

$$F(A)\coloneqq KK^{\mathrm{naive}}(G;A,B) \tag{5.12}$$

on objects. If x is a morphism in $\mathcal{C}(A',A)$, then it can be considered an element of $KK_*^G(A',A)$ naturally and $F(x):F(A')\to F(A)$ is given by the multiplication

$$x\otimes\cdot : KK^G(A(Y),B)\to KK^G(A'(Y),B) \tag{5.13}$$

at the inductive limit level. Then F is a cohomological functor on $\mathcal{C}$. Moreover, using the identity $K_*(A)=KK_*(\mathbb{C},A)$, we get a map

$$\gamma_A : KK_*^{\mathrm{naive}}(G;A,B)\to \mathrm{Hom}^*_{\mathbb{Z}}(K_*(A),KK_*^{\mathrm{naive}}(G;\mathbb{C},B)). \tag{5.14}$$

This is certainly a natural transformation and we need to show that if $K_*(A)$ is finitely generated and free then γ_A is an isomorphism. Using the finite-additivity of both sides, it is enough to consider the cases $A=\mathbb{C}$ and $A=\Sigma\mathbb{C}$, which are obvious.

The last assertion is clear. □

We note that since $KK^{\mathrm{naive}}(G;A,B)$ is not necessarily σ-additive in A, the functor F above is *not* σ-UCT in general.

Applying the Five-Lemma, we get the following.

Corollary 5.7. *Let B be a G-algebra. Suppose that G satisfies the Baum-Connes conjecture for B. Then for any A satisfying the UCT and having finitely generated K-theory, G satisfies the naive Baum-Connes conjecture (2.3) for (A,B).* □

Let A be an algebra satisfying the UCT and having finitely generated K-theory. Then the comparison map $\nu_G^{A,B}$ is an isomorphism. Indeed, by Corollary 5.7, it is enough to show that G satisfies the usual Baum-Connes conjecture for $B \otimes P$. But this is clear since $B \otimes P$ belongs to $\langle \mathscr{C}\mathscr{I} \rangle$ by [MN06, Lemma 4.2] and elements of $\langle \mathscr{C}\mathscr{I} \rangle$ satisfy the usual Baum-Connes conjecture by [MN06, Theorem 5.2].

In particular, the two versions of the mod-n Baum-Connes conjecture are equivalent and they are implied by the usual Baum-Connes conjecture.

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